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The asymptotic formula in Waring's problem: Higher order expansions

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Abstract. When $k > 1$ and s is sufficiently large in terms of k, we derive an explicit multi-term asymptotic expansion for the number of representations of a large natural number as the sum of s positive integral k-th powers.

1. Introduction

As is usual in Waring's problem, when $k > 1$, we let $R_s(n)$ denote the number of representations of n as the sum of s k-th powers of positive integers. Then, as first discovered by Hardy and Littlewood [4], provided that s is sufficiently large in terms of k, one has the asymptotic formula

$$R_s(n) \sim \frac{\Gamma(1+1/k)^s}{\Gamma(s/k)} \mathfrak{S}_s(n) n^{s/k-1} \tag{1.1}$$

as $n \to \infty$, where the *singular series* $\mathfrak{S}_s(n)$ is defined by

$$\mathfrak{S}_s(n) = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left(q^{-1} \sum_{r=1}^{q} e(ar^k/q) \right)^s e(-na/q). \tag{1.2}$$

Here we use the familiar notation $e(z) = e^{2\pi i z}$. This asymptotic formula has been established by the first author [10,11] for $s \geqslant 2^k$ ($k \geqslant 3$), and by the second [14,15] for $s \geqslant 2k^2 - 2k - 8$ ($k \geqslant 6$). We refer the reader to the end of the introduction for an outline of the very latest developments on this topic. Meanwhile, Loh [8] has demonstrated limitations to the quality of the error term which can be obtained in formula (1.1). In this memoir we explain the enigmatic phenomenon discovered by Loh by showing, for the first time, that there are second and higher order terms present in the asymptotic expansion of $R_s(n)$. These new terms resemble the main term, though for odd k there are intriguing differences.

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Suppose that n is a natural number sufficiently large in terms of s and k, and define $P = n^{1/k}$. Let $\mathfrak{M}$ denote the union of the *major arcs*

$$\mathfrak{M}(q,a) = \{\alpha \in [0,1) : |q\alpha - a| \leqslant P/(2kn)\},$$

with $0 \leqslant a \leqslant q \leqslant P$ and $(a,q) = 1$, and define the *minor arcs* $\mathfrak{m}$ by means of the relation $\mathfrak{m} = [0,1) \setminus \mathfrak{M}$. In addition, we introduce the Weyl sum

$$f(\alpha) = \sum_{1 \leqslant x \leqslant P} e(\alpha x^k). \tag{1.3}$$

Finally, when ν is a real number, we say that the exponent t is ν-admissible for k when

$$\int_{\mathfrak{m}} |f(\alpha)|^t \,\mathrm{d}\alpha = o(P^{t-k-\nu}) \tag{1.4}$$

as $P \to \infty$. We note that, given a 0-admissible exponent s_0, the asymptotic formula (1.1) holds whenever $s \geqslant \max\{s_0, 5, k+1\}$ (see [13, Theorem 4.4]).

When $\nu \geqslant 1$, define the exponent $\sigma_\nu(k)$ by

$$\sigma_\nu(k) = \begin{cases} 2^k + 2^{k-1}\nu, & \text{when } 2 \leqslant k \leqslant 5,\\ 2k^2 - 2 + 2^{k-1}(\nu - 1), & \text{when } k = 6, 7,\\ 4k - 2 + 2k(k-2)\nu, & \text{when } k \geqslant 8. \end{cases}$$

Then one may show that the exponent s is ν-admissible for k when $s > \sigma_\nu(k)$. When $2 \leqslant k \leqslant 5$, such follows from the classical approach of [13, Chapters 2 and 4]. Indeed, a careful analysis of the methods underlying [10, 11] (incorporating refinements in [1, 3, 5]) reveals that when $k \geqslant 3$ the exponent $\frac{3}{2}2^k$ is 1-admissible for k, and likewise that 2^{k+1} is 2-admissible for k. When $k \geqslant 8$, on the other hand, the above assertion follows by combining [15, Theorems 10.1 and 11.1], whilst for $k = 6, 7$ one instead combines [15, Theorem 10.1] with Weyl's inequality (see [13, Lemma 2.4]).

In Sections 2–11 we enhance the familiar analysis of the major arc contribution in Waring's problem so as to derive higher order asymptotic expansions of shape

$$R_s(n) = n^{s/k-1}(\mathfrak{C}_0 + \mathfrak{C}_1 n^{-1/k} + \cdots + \mathfrak{C}_J n^{-J/k}) + o(n^{(s-J)/k-1}) \tag{1.5}$$

as $n \to \infty$. We emphasise that our analysis of the minor arc contribution in (1.5) makes use of nothing beyond the bound (1.4) as made available from the above discussion. It is in the analysis of the major arc contribution that our main innovations are to be found. We divide our results according to whether k is even or odd. Here and in what follows, we put $\delta_k = 1$ when $k = 2$, and $\delta_k = 0$ when $k \geqslant 3$.

Theorem 1.1. *Let k be even and $J \geqslant 0$. Suppose that s is J-admissible for k and $s \geqslant (J+1)(k+2) + \delta_k$. Then one has the asymptotic formula* (1.5) *with*

$$\mathfrak{C}_j = \left(-\tfrac{1}{2}\right)^j \binom{s}{j} \frac{\Gamma(1+1/k)^{s-j}}{\Gamma((s-j)/k)} \mathfrak{S}_{s-j}(n) \quad (0 \leqslant j \leqslant J). \tag{1.6}$$

Note that the singular series $\mathfrak{S}_{s-j}(n)$ in this statement is defined via (1.2). We recall that when $s \geqslant 4$, the singular series $\mathfrak{S}_s(n)$ converges absolutely and is non-negative. Further, one has $\mathfrak{S}_s(n) \ll 1$ when $s \geqslant k+2+\delta_k$, and $\mathfrak{S}_s(n) \ll n^\varepsilon$ when $s = k+1+\delta_k$. It is known that when $s \geqslant 4k$, the singular series satisfies the lower bound $\mathfrak{S}_s(n) \gg 1$, and that such remains true for $s \geqslant 5$ when $k=2$, for $s \geqslant 4$ when $k=3$, and for $s \geqslant \frac{3}{2}k$ when k is not a power of 2. This lower bound also holds for the integer n provided that $s \geqslant k+1+\delta_k$ and in addition, for every natural number q, the congruence $x_1^k + \cdots + x_s^k \equiv n \pmod{q}$ possesses a solution with $(x_1, q) = 1$ (see the final sections of [13, Chapters 2 and 4] for an account of such matters).

In order to describe our asymptotic formula when k is odd, we must introduce a modified singular series. When $a \in \mathbb{Z}$ and $q \in \mathbb{N}$, define

$$S(q,a) = \sum_{r=1}^{q} e(ar^k/q) \quad \text{and} \quad T(q,a) = \sum_{r=1}^{q} \left(\frac{1}{2} - \frac{r}{q}\right) e(ar^k/q). \tag{1.7}$$

Then, when $0 \leqslant j \leqslant s$, we define the modified singular series

$$\mathfrak{S}_{s,j}(n) = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} (q^{-1}S(q,a))^{s-j} T(q,a)^j e(-na/q). \tag{1.8}$$

Notice that $\mathfrak{S}_{s,0}(n) = \mathfrak{S}_s(n)$. We demonstrate in Lemma 10.2 that the singular series $\mathfrak{S}_{s,j}(n)$ is absolutely convergent for $s \geqslant \frac{1}{2}(j+2)(k+2)$.

Theorem 1.2. *Let k be odd, $k \geqslant 3$ and $0 \leqslant J \leqslant k$. Suppose that the exponent s is J-admissible for k and $s \geqslant (J+1)(k+2)$. Then one has the asymptotic formula* (1.5) *with*

$$\mathfrak{C}_j = \binom{s}{j} \frac{\Gamma(1+1/k)^{s-j}}{\Gamma((s-j)/k)} \mathfrak{S}_{s,j}(n) \quad (0 \leqslant j \leqslant J). \tag{1.9}$$

Aficionados of the circle method will anticipate that similar conclusions may be obtained for almost all integers n, in the sense of natural density, under weaker conditions on s.

Theorem 1.3. *Let $k \geqslant 2$ and $J \geqslant 0$. Suppose that $2s$ is $2J$-admissible for k and s satisfies $s \geqslant (J+1)(k+2) + \delta_k$. Then one has the following conclusions.*

(i) *When k is even, the asymptotic formula* (1.5) *holds for almost all integers n, with coefficients given by* (1.6).

(ii) *When k is odd and $J \leqslant k$, the asymptotic formula* (1.5) *holds for almost all integers n, with coefficients given by* (1.9).

As we have noted, the main term in the asymptotic formula (1.5) is classical. This much was established by Hardy and Littlewood [4] in their series of seminal papers concerning the application of their circle method to Waring's problem. Beyond this main term little was known until the work of Loh [8]. This shows that when $k \geqslant 3$, one has

$$R_{k+1}(n) - \Gamma(1+1/k)^k \mathfrak{S}_{k+1}(n) n^{1/k} = \Omega(n^{1/(2k)}),$$

and further that for $s \geqslant k+2$, one has

$$R_s(n) - \frac{\Gamma(1+1/k)^s}{\Gamma(s/k)} \mathfrak{S}_s(n) n^{s/k-1} = \Omega_-(n^{(s-1)/k-1}). \tag{1.10}$$

Theorem 1.1 shows that for even k, the Ω_--result (1.10) is explained precisely by the presence in the asymptotic formula (1.5) of the secondary term

$$\mathfrak{C}_1 n^{(s-1)/k-1} = -\tfrac{1}{2} s \frac{\Gamma(1+1/k)^{s-1}}{\Gamma((s-1)/k)} \mathfrak{S}_{s-1}(n) n^{(s-1)/k-1}.$$

When k is odd, Theorem 1.2 shows instead that one has the secondary term

$$\mathfrak{C}_1 n^{(s-1)/k-1} = s \frac{\Gamma(1+1/k)^{s-1}}{\Gamma((s-1)/k)} \mathfrak{S}_{s,1}(n) n^{(s-1)/k-1}.$$

Presumably, the modified singular series $\mathfrak{S}_{s,1}(n)$ is non-zero under modest conditions, and this would again precisely explain Loh's discovery. However, when k is odd this series does not have an interpretation as an Euler product, and so in general it is not entirely clear how it behaves. In Section 13 we explore what can be said concerning the behaviour of $\mathfrak{S}_{s,j}(n)$.

Theorem 1.4. *Suppose that k is odd and $s \geqslant \frac{3}{2}k + 3$. Let Q be a positive integer, and let n be a multiple of $Q!$. Then one has*

$$\mathfrak{S}_{s,1}(n) = -\tfrac{1}{2}\mathfrak{S}_{s-1}(n) + O(Q^{-1/(2k)}).$$

When k is odd and $s \geqslant \frac{3}{2}k + 3$, the singular series $\mathfrak{S}_{s-1}(n)$ is positive and bounded away from zero (see [13, Theorem 4.6]). By taking $Q = Q(s,k)$ to be sufficiently large in terms of s and k, it therefore follows that $-\mathfrak{S}_{s,1}(n) \gg 1$ for a positive proportion of n. If instead one takes Q to grow slowly with n, say $Q = \sqrt{\log\log n}$, one has $Q! = o(\log n)$, and hence there are at least $N/\log N$ integers with $1 \leqslant n \leqslant N$ for which

$$\mathfrak{S}_{s,1}(n) = -\tfrac{1}{2}\mathfrak{S}_{s-1}(n) + O(1/\log\log\log N).$$

Thus $\mathfrak{S}_{s,1}(n)$ is frequently very close to $-\frac{1}{2}\mathfrak{S}_{s-1}(n)$. One is tempted to believe that in fact this is usually the case, but in any case Loh's conclusion (1.10) is explained by this observation for odd k.

Finally, we show that the modified singular series $\mathfrak{S}_{s,j}(n)$ is often non-zero for small values of j.

Theorem 1.5. *Suppose that $j \geqslant 0$ and $s \geqslant \frac{1}{2}(j+4)(k+2)$. Then there is a constant $C_j > 0$ such that, for all sufficiently large x, the number $N_j(x)$ of integers n with $1 \leqslant n \leqslant x$ for which $|\mathfrak{S}_{s,j}(n)| \geqslant C_j$ satisfies $N_j(x) \geqslant C_j x$.*

The reader having a passing familiarity with the theory of modular forms will recognise that in the case $k = 2$, corresponding to the representation of integers as sums of squares, very precise asymptotic formulæ are available involving the Fourier coefficients of Eisenstein series and cusp forms (see, for example, [6, Section 11.3]). This observation might prompt speculation that some exotic generalisation of Eisenstein series and cusp forms might conceivably describe $R_s(n)$ also when $k \geqslant 3$. When k is even, the asymptotic formula (1.5) supplied by Theorem 1.1 seems consistent with this speculation, since each term is given by a classical singular series having an Euler product interpretation. When k is odd, however, the modified singular series $\mathfrak{S}_{s,j}(n)$ pose interesting problems for such an explanation. Perhaps the exponential sums

$$\sum_{r=1}^{q} \psi(r/q) e(ar^k/q),$$

in which $\psi \in \mathbb{Z}[x]$ has positive degree, demand further investigation.

This paper is organised as follows. In Sections 2–6 we examine even k. Following some preliminary discussion in Section 2, we establish basic major arc estimates in Section 3. Certain auxiliary estimates require multi-term asymptotic expansions, and so in Section 4 we apply Euler–MacLaurin expansions, inserting the output into corresponding major arc estimates in Section 5. We complete the proof of Theorem 1.1 in Section 6 by combining the contributions of these estimates. The treatment of odd k in Sections 7–11 mirrors that of even k, though in Section 10 we briefly discuss the novel modified singular series $\mathfrak{S}_{s,j}(n)$. This establishes Theorem 1.2. In Section 12 we discuss exceptional sets, proving Theorem 1.3. Finally, in Section 13, we investigate the singular series $\mathfrak{S}_{s,j}(n)$ for odd k, establishing Theorems 1.4 and 1.5.

Our basic parameter is P, a sufficiently large positive number, and we will normally take $P = n^{1/k}$. Exceptionally in Section 12 we will take $P = N^{1/k}$. In the o-notation the limiting process will invariably be as $P \to \infty$, or equivalently n or $N \to \infty$. In this paper, implicit constants in Vinogradov's notation $\ll$ and $\gg$ may depend on s, k and ε. Whenever ε appears in a statement, either implicitly or explicitly, we assert that the statement holds for each $\varepsilon > 0$. Finally, we write $\|\theta\| = \min_{m\in\mathbb{Z}} |\theta - m|$.

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Recent work on the asymptotic formula in Waring's problem. The number of variables required to establish the anticipated asymptotic formula in Waring's problem depends, of course, on available estimates for mean values of Weyl sums. This is a topic which has very recently been subject to rapid advances. Consequently, in the above discussion, we have retained an account of the status of the topic at the time of submission of this paper, 30th August 2013. Let $\widetilde{G}(k)$ denote the least positive integer t having the property that, whenever $s \geqslant t$, then the asymptotic formula (1.1) holds. Then at the time of this revision (20th October 2015), it is known that $\widetilde{G}(3) \leqslant 8$, $\widetilde{G}(4) \leqslant 16$ (see [10, 11]), $\widetilde{G}(5) \leqslant 28$, $\widetilde{G}(6) \leqslant 43$, $\widetilde{G}(7) \leqslant 61$, $\widetilde{G}(8) \leqslant 83$, and so on (see [18, Theorem 1.4]) and $\widetilde{G}(k) \leqslant (C + o(1))k^2$ with $C = 1.540789\ldots$ (see [16, Theorem 12.2]). In addition, it follows from the latter sources together with [17, Theorem 1.2] that s is ν-admissible for k when $s > \sigma_\nu(k)$, where

$$\sigma_\nu(k) = \begin{cases} 2^k + 2^{k-1}\nu, & \text{when } k = 2, 3,\\ 2k(k-1) + 2^{k-1}(\nu - 1), & \text{when } 4 \leqslant k \leqslant 6,\\ 4k - 4 + (2k^2 - 6k + 4)\nu, & \text{when } k \geqslant 7. \end{cases}$$

Indeed, some modest refinements can be made available for $k \geqslant 7$ (see [18, Theorem 11.5]), and for large values of k one may take

$$\sigma_\nu(k) = 2k(k-1) + (\nu - 1)(2k^2 - k^{3/2}/\sqrt{3} + O(k))$$

(combine [18, Theorem 11.7] and [17, Theorem 1.2]). Further such advances appear imminent.

2. Preliminary manœuvres, for even k

Suppose that s and k are natural numbers with $s > k \geqslant 2$ and k even. We establish the multi-term asymptotic formula claimed in Theorem 1.1 by applying the Hardy–Littlewood method to analyse a modification of the standard Waring problem. Let n be a positive integer sufficiently large in terms of s and k. We recall that $P = n^{1/k}$, and define $R_s^*(n)$ to be the number of integral representations of n in the shape

$$n = x_1^k + \cdots + x_s^k, \tag{2.1}$$

with $|x_i| \leqslant P$ $(1 \leqslant i \leqslant s)$. It is apparent that $R_s^*(n)$ is approximately $2^s R_s(n)$. On accounting for the contribution arising from those representations in which one or more variables are zero, we find that

$$R_s^*(n) = \sum_{r=0}^{s} 2^{s-r} \binom{s}{r} R_{s-r}(n), \tag{2.2}$$

whence

$$R_s(n) = 2^{-s} \sum_{r=0}^{s} (-1)^r \binom{s}{r} R_{s-r}^*(n). \tag{2.3}$$

Indeed, on substituting (2.2) into the right-hand side of (2.3), we see that

$$\begin{aligned} 2^{-s} \sum_{r=0}^{s} (-1)^r \binom{s}{r} R_{s-r}^*(n) &= 2^{-s} \sum_{r=0}^{s} (-1)^r \binom{s}{r} \sum_{l=0}^{s-r} 2^{s-r-l} \binom{s-r}{l} R_{s-r-l}(n) \\ &= \sum_{u=0}^{s} 2^{-u} \binom{s}{u} R_{s-u}(n) \sum_{l=0}^{u} (-1)^{u-l} \binom{u}{l}. \end{aligned}$$

The innermost sum on the right-hand side is equal to $(1-1)^u$, and so the only non-zero term in the outermost sum is that with $u = 0$. The claimed relation (2.3) therefore follows. In order to establish Theorem 1.1, it suffices to obtain a multi-term asymptotic expansion for $R_t^*(n)$ when t is close to s (in fact, when $s - t \leqslant k + J$). This, it transpires, is more easily accomplished than the analogous task for $R_t(n)$.

Next, define the generating function

$$h(\alpha) = \sum_{|x| \leqslant P} e(\alpha x^k), \tag{2.4}$$

and, when $\mathfrak{B} \subseteq [0, 1)$ is measurable, put

$$R_t^*(n; \mathfrak{B}) = \int_{\mathfrak{B}} h(\alpha)^t e(-n\alpha)\, d\alpha. \tag{2.5}$$

By orthogonality, we have

$$R_t^*(n) = R_t^*(n; \mathfrak{M}) + R_t^*(n; \mathfrak{m}). \tag{2.6}$$

The hypotheses of the statement of Theorem 1.1 permit us the assumption that s is J-admissible for k, and this, in essence, takes care of the analysis of $R_t^*(n; \mathfrak{m})$. The lemma below formalises this observation. We note for future reference that, in view of (1.3) and (2.4), one has the relation

$$h(\alpha) = 1 + 2f(\alpha). \tag{2.7}$$

Lemma 2.1. *Suppose that $J \geqslant 0$ and $s \geqslant k + J + 1$. Then for $0 \leqslant t \leqslant s - k - J - 1$, one has*

$$R_t^*(n; [0, 1)) \ll P^{s-k-J-1}. \tag{2.8}$$

Suppose instead that s is J-admissible for k and $s-k-J \leqslant t \leqslant s$. Then

$$R_t^*(n;\mathfrak{m}) = o(P^{s-k-J}). \tag{2.9}$$

Proof. An application of the triangle inequality within (2.5) leads, via (2.7), to the bound

$$R_t^*(n;\mathfrak{B}) \ll 1 + \int_{\mathfrak{B}} |f(\alpha)|^t \,\mathrm{d}\alpha.$$

When $0 \leqslant t \leqslant s-k-J-1$, therefore, the trivial bound $|f(\alpha)| \leqslant P$ yields (2.8). When instead $s-k-J \leqslant t \leqslant s$, one finds from Hölder's inequality that

$$R_t^*(n;\mathfrak{m}) \ll 1 + \left(\int_{\mathfrak{m}} |f(\alpha)|^s \,\mathrm{d}\alpha \right)^{t/s} \left(\int_0^1 \mathrm{d}\alpha \right)^{1-t/s}.$$

In the first integral on the right-hand side, we invoke the hypothesis that s is J-admissible for k, and apply the associated estimate (1.4) with t replaced by s. Since $t/s \leqslant 1$, we obtain (2.9). This completes the proof of the lemma. □

By substituting the conclusions of Lemma 2.1 into (2.3), and noting (2.6), we deduce that when s is J-admissible and $s \geqslant k+J+1$, then

$$R_s(n) = 2^{-s} \sum_{r=0}^{k+J} (-1)^r \binom{s}{r} R_{s-r}^*(n;\mathfrak{M}) + o(P^{s-k-J}). \tag{2.10}$$

Thus it remains to analyse $R_{s-r}^*(n;\mathfrak{M})$ for $0 \leqslant r \leqslant k+J$.

3. The major arc contribution truncated, for even k

Our first step in the analysis of $R_{s-r}^*(n;\mathfrak{M})$ is the replacement of the generating function $h(\alpha)$ in (2.5) by a suitable approximation. This requires a little preparation. Define $S(q,a)$ as in (1.7), and when $\beta \in \mathbb{R}$ put

$$v(\beta) = \int_0^P e(\beta\gamma^k)\,\mathrm{d}\gamma. \tag{3.1}$$

We define $f^*(\alpha)$ for $\alpha \in \mathfrak{M}$ by taking

$$f^*(\alpha) = q^{-1}S(q,a)v(\alpha - a/q) \quad \text{when } \alpha \in \mathfrak{M}(q,a). \tag{3.2}$$

From [13, Theorem 4.1], it therefore follows that when $0 \leqslant a \leqslant q \leqslant P$ and $(a,q)=1$, one has

$$\sup_{\alpha \in \mathfrak{M}(q,a)} |f(\alpha) - f^*(\alpha)| \ll q^{1/2+\varepsilon} \leqslant P^{1/2+\varepsilon}, \tag{3.3}$$

whence (2.7) yields

$$\sup_{\alpha \in \mathfrak{M}} |h(\alpha) - 2f^*(\alpha)| \ll P^{1/2+\varepsilon}. \tag{3.4}$$

An application of the binomial theorem within (2.5) reveals that for non-negative integers t, one has

$$R_t^*(n;\mathfrak{M}) = \sum_{l=0}^{t} \binom{t}{l} \mathfrak{I}_{t,l}(n), \tag{3.5}$$

where

$$\mathfrak{I}_{t,l}(n) = \int_{\mathfrak{M}} (2f^*(\alpha))^{t-l}(h(\alpha)-2f^*(\alpha))^l e(-n\alpha)\,\mathrm{d}\alpha. \tag{3.6}$$

The motivation for this application of the binomial theorem, and also a subsequent one underlying the proof of Lemma 6.1, is to isolate parts of the quantity $R_t^*(n;\mathfrak{M})$ that may be conveniently truncated and absorbed into the error term. The naive reader might be puzzled by the apparently complicated applications of the binomial theorem forwards and backwards in the discussion to come. The reason that one cannot simply read off a suitable asymptotic formula for the major arc contribution $R_t^*(n;\mathfrak{M})$ lies in the observation that, although we have good control of the error term in the approximation of $f(\alpha)$ by $f^*(\alpha)$ for $\alpha \in \mathfrak{M}$, the corresponding error in approximating $f(\alpha)^t$ by $f^*(\alpha)^t$ is complicated by the presence of numerous cross terms. Meanwhile, t-dimensional approaches to approximating $f(\alpha)^t$ directly by $f^*(\alpha)^t$ are not available in the absence of smoothing weights, the presence of which would sabotage our asymptotic formula.

Lemma 3.1. *Suppose that $k \geqslant 2$, and that J and r are non-negative integers. Then whenever $l > 2J - 2r$ and $s \geqslant \max\{l+r, k+2J+4\}$, one has*

$$\mathfrak{I}_{s-r,l}(n) = o(P^{s-k-J}).$$

Proof. When $k = 2$, the methods of [13, Chapter 4] deliver the upper bound

$$\int_{\mathfrak{M}} |f^*(\alpha)|^{k+2}\,\mathrm{d}\alpha \ll P^{2+\varepsilon}, \tag{3.7}$$

a bound that may be confirmed also when $k \geqslant 3$ by the methods underlying the proof of [12, Lemma 5.1]. We apply this estimate in order to simplify the estimation of the integral $\mathfrak{I}_{s-r,l}(n)$.

We begin by considering the situation in which $s \geqslant k+r+l+2$. Here, by applying the trivial bound $f^*(\alpha) \ll P$ in (3.6), and then utilising (3.4) and (3.7), we obtain the estimate

$$\begin{aligned}\mathfrak{I}_{s-r,l}(n) &\ll \Big(\sup_{\alpha\in\mathfrak{M}} |h(\alpha)-2f^*(\alpha)|\Big)^l \Big(P^{s-k-r-l-2}\int_{\mathfrak{M}} |f^*(\alpha)|^{k+2}\,\mathrm{d}\alpha\Big)\\ &\ll P^{s-k-r-l/2+\varepsilon}.\end{aligned}$$

The hypothesis $l > 2J - 2r$ therefore ensures that

$$\mathfrak{I}_{s-r,l}(n) \ll P^{s-k-J-1/2+\varepsilon} = o(P^{s-k-J}).$$

This completes the proof of the lemma in this first situation.

It remains to consider those circumstances in which

$$r + l \leqslant s \leqslant k+r+l+1.$$

Here we may assume without loss that $l \geqslant 2J - 2r + 3$, for if instead one were to have $l \leqslant 2J - 2r + 2$, then

$$s \leqslant k + r + (2J - 2r + 2) + 1 \leqslant k + 2J + 3,$$

contradicting the hypothesis $s \geqslant k + 2J + 4$. Note next that the measure of $\mathfrak{M}$ is $O(P^{2-k})$. Let $\omega = (s - r - l)/(k + 2)$. Then, by applying Hölder's inequality to (3.6), we obtain

$$\mathfrak{I}_{s-r,l}(n) \ll \Big(\sup_{\alpha \in \mathfrak{M}} |h(\alpha) - 2f^*(\alpha)| \Big)^l \Big(\int_{\mathfrak{M}} |f^*(\alpha)|^{k+2} \,\mathrm{d}\alpha \Big)^{\omega} \Big(\int_{\mathfrak{M}} \mathrm{d}\alpha \Big)^{1-\omega}.$$

We therefore find from (3.4) and (3.7) that $\mathfrak{I}_{s-r,l}(n) = O(P^{\lambda+\varepsilon})$, where

$$\lambda = s - k - r - \tfrac{1}{2} l + 2 - 2(s - r - l)/(k + 2). \tag{3.8}$$

We now divide into cases.

First, when $l > 2J - 2r + 4$, it follows from the hypothesis $s \geqslant l + r$ that

$$l > 2J - 2r + 4 - 4(s - r - l)/(k + 2),$$

whence

$$r + \tfrac{1}{2} l - 2 + 2(s - r - l)/(k + 2) > J.$$

Thus we deduce from (3.8) that $\lambda < s - k - J$, so that $\mathfrak{I}_{s-r,l}(n) = o(P^{s-k-J})$.

Otherwise, we have

$$2J - 2r + 3 \leqslant l \leqslant 2J - 2r + 4.$$

Here, if we were to have $s \leqslant \frac{1}{4}(k + 2) + r + l$, then we obtain

$$s \leqslant k + 2J - r + \tfrac{1}{4}(18 - 3k) \leqslant k + 2J + 3,$$

contradicting the hypothesis $s \geqslant k + 2J + 4$. Then we have $s > \frac{1}{4}(k + 2) + r + l$, so from (3.8) we infer that

$$\lambda < s - k - r - \tfrac{1}{2}(l - 3) \leqslant s - k - r - (J - r).$$

Thus, in this final situation, we again deduce that $\mathfrak{I}_{s-r,l}(n) = o(P^{s-k-J})$, and the proof of the lemma is complete. □

Notice that when $s \geqslant k + 2J + 4$ and $r > J$, the conclusion of Lemma 3.1 ensures that $\mathfrak{I}_{s-r,l}(n) = o(P^{s-k-J})$. Thus, on combining (2.10) and (3.5) with Lemma 3.1, we deduce that whenever $s \geqslant k + 2J + 4$ one has

$$R_s(n) = 2^{-s} \sum_{r=0}^{J} (-1)^r \binom{s}{r} \sum_{l=0}^{2J-2r} \binom{s-r}{l} \mathfrak{I}_{s-r,l}(n) + o(P^{s-k-J}). \tag{3.9}$$

4. An auxiliary lemma, for even *k*

Before proceeding further, we must estimate certain multiple sums over arithmetic progressions. We first recall two standard tools, beginning with the Euler–MacLaurin summation formula. The associated Bernoulli numbers B_κ ($\kappa \geqslant 0$) may be defined by putting $B_0 = 1$, $B_1 = -\frac{1}{2}$, and iterating the relation

$$B_\kappa = \sum_{j=0}^{\kappa} \binom{\kappa}{j} B_{\kappa-j} \quad (\kappa \geqslant 2).$$

The Bernoulli polynomials $B_\kappa(x)$ may then be defined by taking

$$B_\kappa(x) = \sum_{j=0}^{\kappa} \binom{\kappa}{j} B_{\kappa-j} x^j \quad (\kappa \geqslant 0).$$

We write $\{x\} = x - \lfloor x \rfloor$, where $\lfloor x \rfloor$ denotes the greatest integer no larger than x, and write $\lceil x \rceil$ for the least integer no smaller than x. It is convenient then to write $\beta_\kappa(x) = B_\kappa(\{x\})$ for $\kappa \geqslant 0$.

Lemma 4.1. *Let a and b be real numbers with $a < b$, and let K be a positive integer. Suppose that F has continuous derivatives through the $(K-1)$-st order on $[a,b]$, that the K-th derivative of F exists and is continuous on (a,b), and $|F^{(K)}(x)|$ is integrable on $[a,b]$. Then*

$$\sum_{a<n\leqslant b} F(n) = \int_a^b F(x)\,\mathrm{d}x + \sum_{\kappa=1}^{K} \frac{(-1)^\kappa}{\kappa!}\big(\beta_\kappa(b)F^{(\kappa-1)}(b) - \beta_\kappa(a)F^{(\kappa-1)}(a)\big) - \frac{(-1)^K}{K!}\int_a^b \beta_K(x)F^{(K)}(x)\,\mathrm{d}x.$$

Proof. This is essentially the version of the Euler–MacLaurin summation formula provided in [9, Theorem B.5]. The statement of the latter demands that $F^{(K)}(x)$ exist and be continuous on $[a,b]$. However, the argument of the proof of [9, Theorem B.5] remains applicable if instead $F^{(K)}(x)$ exists and is continuous on (a,b), and in addition $|F^{(K)}(x)|$ is integrable on $[a,b]$. □

Next, we recall Faà di Bruno's formula for the N-th derivative of a composition of functions.

Lemma 4.2. *Suppose that F and G have continuous derivatives of order up to the N-th on an open interval containing x. Then*

$$\frac{\mathrm{d}^N}{\mathrm{d}x^N} F(G(x)) = \sum \frac{N!}{m_1!\dots m_N!} F^{(m_1+\dots+m_N)}(G(x)) \prod_{j=1}^{N} \left(\frac{G^{(j)}(x)}{j!}\right)^{m_j},$$

where the summation is over non-negative integers $m_1, \dots, m_N$ satisfying

$$m_1 + 2m_2 + \dots + Nm_N = N.$$

Proof. See [7] for an account of this formula and its history. □

We apply Lemmata 4.1 and 4.2 in combination to obtain an asymptotic formula for an important auxiliary sum. Let X be a positive real number, and let θ be a non-negative real exponent. When $q \in \mathbb{N}$ and $r \in \mathbb{Z}$, we define

$$\Upsilon_{q,r}(X;\theta) = \sum_{-(X+r)/q \leqslant h \leqslant (X-r)/q} (X^k - (qh+r)^k)^\theta. \tag{4.1}$$

We shall encounter the expression $\Upsilon_{q,r}(X;\theta)$ in the discussion to come only when θ is an integral multiple of $1/k$.

Lemma 4.3. *When $1 \leqslant N \leqslant \lceil \theta \rceil$, one has*

$$\Upsilon_{q,r}(X;\theta) = (2X/q)X^{k\theta} \frac{\Gamma(1+\theta)\Gamma(1+1/k)}{\Gamma(1+\theta+1/k)} + O\big(X^{k\theta}(q/X)^{N-1}\big).$$

Proof. We apply Lemma 4.2 with

$$F(y) = y^\theta \quad \text{and} \quad G(x) = X^k - (qx+r)^k. \tag{4.2}$$

Write $a = -(X+r)/q$ and $b = (X-r)/q$. Then one finds that

$$G(a) = X^k - (-X)^k = 0 \quad \text{and} \quad G(b) = X^k - X^k = 0.$$

Further, when $0 \leqslant j \leqslant k$, one has

$$G^{(j)}(x) = -\frac{k!}{(k-j)!} q^j (qx+r)^{k-j}, \tag{4.3}$$

whilst $G^{(j)}(x) = 0$ for $j > k$. Also, when $0 \leqslant m \leqslant \lceil \theta \rceil$, one has

$$F^{(m)}(y) = \theta(\theta-1)\dots(\theta-m+1)y^{\theta-m}, \tag{4.4}$$

where the condition $y \neq 0$ should be imposed in case $m > \theta$. It follows from Lemma 4.2 that $F(G(x))$ has continuous derivatives through the N-th order on (a,b), continuous derivatives through the $(N-1)$-st order on $[a,b]$, and further $|\mathrm{d}^N F(G(x))/\mathrm{d}x^N|$ is integrable on $[a,b]$. Note also that when $0 \leqslant m < \theta$, one has $F^{(m)}(G(a)) = F^{(m)}(G(b)) = 0$. Then Lemma 4.2 shows that

$$\frac{\mathrm{d}^\kappa}{\mathrm{d}x^\kappa} F(G(x))\Big|_{x=a} = \frac{\mathrm{d}^\kappa}{\mathrm{d}x^\kappa} F(G(x))\Big|_{x=b} = 0 \quad (0 \leqslant \kappa < \theta).$$

On substituting these conclusions into Lemma 4.1, we see that

$$\sum_{a \leqslant h \leqslant b} F(G(h)) = \int_a^b F(G(x))\,\mathrm{d}x - \frac{(-1)^N}{N!} \int_a^b \beta_N(x) \frac{\mathrm{d}^N}{\mathrm{d}x^N} F(G(x))\,\mathrm{d}x. \tag{4.5}$$

The first term on the right-hand side of (4.5) is easily evaluated. By making the change of variable $y = (qx+r)/X$, we find that

$$\begin{aligned} \int_a^b F(G(x))\,\mathrm{d}x &= q^{-1}X^{k\theta+1} \int_{-1}^1 (1-y^k)^\theta\,\mathrm{d}y \\ &= (2X/q)X^{k\theta} \frac{\Gamma(1+\theta)\Gamma(1+1/k)}{\Gamma(1+\theta+1/k)}. \end{aligned} \tag{4.6}$$

For the second term we must work harder. When θ is an integer, it follows from (4.3) and (4.4) via Lemma 4.2 that one has the upper bound

$$\sup_{a \leqslant x \leqslant b} \left| \frac{\mathrm{d}^N}{\mathrm{d}x^N} F(G(x)) \right| \ll q^N X^{k\theta - N}.$$

When θ is not an integer, on the other hand, say $\{\theta\} = 1 - \nu$, then we find in like manner that

$$\sup_{(a+b)/2 \leqslant x \leqslant b} \left| (X - qx - r)^{\nu} \frac{\mathrm{d}^N}{\mathrm{d}x^N} F(G(x)) \right| \ll q^N X^{k\theta - N + \nu}$$

and

$$\sup_{a \leqslant x \leqslant (a+b)/2} \left| (X + qx + r)^{\nu} \frac{\mathrm{d}^N}{\mathrm{d}x^N} F(G(x)) \right| \ll q^N X^{k\theta - N + \nu}.$$

Thus we deduce that

$$\begin{aligned} (4.7) \qquad & \int_a^b \beta_N(x) \frac{\mathrm{d}^N}{\mathrm{d}x^N} F(G(x))\, \mathrm{d}x \\ & \ll q^N X^{k\theta - N + \nu} \left(\int_a^b (X + qx + r)^{-\nu} + (X - qx - r)^{-\nu}\, \mathrm{d}x \right) \\ & \ll (q^N X^{k\theta - N + \nu})(q^{-1} X^{1-\nu}) \ll X^{k\theta} (q/X)^{N-1}. \end{aligned}$$

Since this estimate is immediate when θ is an integer, the conclusion of the lemma follows on substituting (4.6) and (4.7) into (4.5), and then recalling the definition (4.1) of $\Upsilon_{q,r}(X;\theta)$. □

This lemma may be extended by induction to derive a multidimensional generalisation. When $q \in \mathbb{N}$ and $r_1, \dots, r_l \in \mathbb{Z}$, we define

$$(4.8) \qquad \Xi_{q,\mathbf{r}}^{(l)}(X;\theta) = \sum_{\substack{|x_1| \leqslant X \\ x_1 \equiv r_1 \pmod q}} \cdots \sum_{\substack{|x_l| \leqslant X \\ x_l \equiv r_l \pmod q}} (X^k - x_1^k - \cdots - x_l^k)^{\theta},$$

where the summands are constrained by the inequality $x_1^k + \cdots + x_l^k \leqslant X^k$.

Lemma 4.4. *When* $1 \leqslant N \leqslant \lceil \theta \rceil$*, one has*

$$\Xi_{q,\mathbf{r}}^{(l)}(X;\theta) = (2X/q)^l X^{k\theta} \frac{\Gamma(1+\theta)\Gamma(1+1/k)^l}{\Gamma(1+\theta+l/k)} + O\big(X^{k\theta}(q/X)^{N-1}(1+X/q)^{l-1}\big).$$

Proof. We proceed by induction on l, noting that the case $l = 1$ is already established by Lemma 4.3. Suppose then that $L > 1$, and that the desired conclusion has been established for $1 \leqslant l < L$. From (4.8), we obtain

$$(4.9) \qquad \Xi_{q,\mathbf{r}}^{(L)}(X;\theta) = \sum_{-(X+r_L)/q \leqslant h_L \leqslant (X-r_L)/q} \Xi_{q,\mathbf{r}'}^{(L-1)}(Y;\theta),$$

in which we have written

$$\mathbf{r}' = (r_1, \dots, r_{L-1}) \quad \text{and} \quad Y = (X^k - (qh_L + r_L)^k)^{1/k}.$$

Our inductive hypothesis supplies the asymptotic formula

$$\begin{aligned} \Xi_{q,\mathbf{r}'}^{(L-1)}(Y;\theta) = (2Y/q)^{L-1} Y^{k\theta} & \frac{\Gamma(1+\theta)\Gamma(1+1/k)^{L-1}}{\Gamma(1+\theta+(L-1)/k)} \\ & + O\big(Y^{k\theta}(q/Y)^{N-1}(1+Y/q)^{L-2}\big). \end{aligned}$$

By substituting this expression into (4.9), we deduce that

$$\Xi_{q,\mathbf{r}}^{(L)}(X;\theta) = \frac{\Gamma(1+\theta)\Gamma(1+1/k)^{L-1}}{\Gamma(1+\theta+(L-1)/k)} T_0 + O\big(X^{k\theta}(q/X)^{N-1}(1+X/q)^{L-1}\big), \tag{4.10}$$

where

$$T_0 = (2/q)^{L-1} \sum_{-(X+r_L)/q \leqslant h_L \leqslant (X-r_L)/q} \big(X^k - (qh_L + r_L)^k\big)^{\theta+(L-1)/k}. \tag{4.11}$$

An application of Lemma 4.3 leads from (4.11) to the asymptotic relation

$$T_0 = (2X/q)^L X^{k\theta} \frac{\Gamma(1+\theta+(L-1)/k)\Gamma(1+1/k)}{\Gamma(1+\theta+L/k)} + O\big(X^{k\theta}(q/X)^{N-L}\big).$$

We therefore infer from (4.10) that the inductive hypothesis holds for $l = L$, confirming the inductive step and completing the proof of the lemma. □

We remark that in the analysis yielding Lemmata 4.3 and 4.4, it is the vanishing of high order derivatives that eliminates the potential existence of additional terms in the asymptotic formula delivered by Theorem 1.1. In circumstances in which θ is an integer, one may apply the Euler–MacLaurin summation formula to obtain additional terms in Lemma 4.3 when $N > \lceil\theta\rceil$, and presumably these would lead to a zoo of additional terms of order $n^{(s-J)/k-1}$ in the asymptotic formula for $R_s(n)$ when s is a multiple of k and $J > \lceil s/k - 1\rceil$.

5. The major arc contribution evaluated, for even k

Our goal in this section is the evaluation of the integral $\mathfrak{I}_{t,l}(n)$ defined in (3.6). This we achieve via a binomial expansion of the term $(h(\alpha) - 2f^*(\alpha))^l$ lying within the integrand. We consequently consider the auxiliary integral

$$\mathfrak{K}_{u,l}(n) = \int_{\mathfrak{M}} (2f^*(\alpha))^u h(\alpha)^l e(-n\alpha)\,\mathrm{d}\alpha. \tag{5.1}$$

Note that, on recalling the definition (2.4) of $h(\alpha)$, this integral may be rewritten in the shape

$$\mathfrak{K}_{u,l}(n) = 2^u \sum_{|m_1|\leqslant P} \cdots \sum_{|m_l|\leqslant P} \mathfrak{R}_u(n - m_1^k - \cdots - m_l^k), \tag{5.2}$$

where

$$\mathfrak{R}_u(m) = \int_{\mathfrak{M}} f^*(\alpha)^u e(-m\alpha)\,\mathrm{d}\alpha. \tag{5.3}$$

It is the presence here of the factor $f^*(\alpha)^u$ in the integrand that is helpful to us. We have been careful to ensure that the exponent u may be taken relatively large in our applications, and this yields much stronger control of the error terms in our asymptotic formula for $\mathfrak{K}_{u,l}(n)$ than would be the case if $f^*(\alpha)$ were to be replaced by $f(\alpha)$, as in classical analyses.

Before refining the conventional major arc analysis of $\mathfrak{R}_u(m)$ so as to extract a sharper error term, we pause to record two estimates for the auxiliary sum

$$V_A^B(u;\theta) = \sum_{A\leqslant q<B} \sum_{\substack{a=1\\(a,q)=1}}^{q} q^\theta |q^{-1}S(q,a)|^u. \tag{5.4}$$

Lemma 5.1. *When $u > k+1+\delta_k$, one has*

$$V_1^Q(u;\theta) \ll 1 + Q^{1+\theta-(u-1-\delta_k)/k+\varepsilon},$$

and when $u > k(1+\theta)+1+\delta_k$, one has

$$V_Q^\infty(u;\theta) \ll Q^{1+\theta-(u-1-\delta_k)/k+\varepsilon}.$$

Proof. The conclusion of [13, Lemma 4.9] supplies the bound

$$\sum_{1\leqslant q\leqslant 2Q} \sum_{\substack{a=1\\(a,q)=1}}^{q} |q^{-1}S(q,a)|^{k+1+\delta_k} \ll Q^\varepsilon.$$

Then it follows from [13, Theorem 4.2] that

$$\sum_{Q\leqslant q<2Q} q^\theta \sum_{\substack{a=1\\(a,q)=1}}^{q} |q^{-1}S(q,a)|^u \ll Q^{1+\theta-(u-1-\delta_k)/k+\varepsilon},$$

and the desired estimates follow by summing over dyadic intervals. □

Before announcing our refinement of the conventional major arc analysis, we define for future reference the truncated singular series

$$\mathfrak{S}_u(m;P) = \sum_{1\leqslant q\leqslant P} \sum_{\substack{a=1\\(a,q)=1}}^{q} (q^{-1}S(q,a))^u e(-ma/q). \tag{5.5}$$

Lemma 5.2. *Suppose that u is an integer with $u \geqslant (J+1)k+2+\delta_k$. Then*

$$\mathfrak{S}_u(m;P) = \mathfrak{S}_u(m) + O\big(P^{-J-1/(2k)}\big).$$

Also, there is a positive number η such that, whenever $|m| \leqslant un$, one has

$$\mathfrak{R}_u(m) = \Delta_m \frac{\Gamma(1+1/k)^u}{\Gamma(u/k)} \mathfrak{S}_u(m) m^{u/k-1} + O\big(P^{u-k-J-\eta}\big),$$

where $\Delta_m = 1$ when $m \geqslant 0$, and $\Delta_m = 0$ when $m < 0$.

Proof. On recalling (3.2) and (5.3), we see that

$$\mathfrak{R}_u(m) = \sum_{1\leqslant q\leqslant P} \sum_{\substack{a=1\\(a,q)=1}}^{q} (q^{-1}S(q,a))^u e(-ma/q) I_u(m;q), \tag{5.6}$$

where

$$I_u(m;q) = \int_{-P/(2kqn)}^{P/(2kqn)} v(\beta)^u e(-\beta m)\,\mathrm{d}\beta.$$

Define

$$I(m) = \int_{-\infty}^{\infty} v(\beta)^u e(-\beta m)\,\mathrm{d}\beta. \tag{5.7}$$

This integral is absolutely convergent for $u \geqslant k+1$, as is immediate from [13, Theorem 7.3]. The latter theorem also yields the estimate

$$I_u(m;q) - I(m) \ll P^u \int_{P/(2kqn)}^{\infty} (1+P^k\beta)^{-u/k}\,\mathrm{d}\beta \ll (qn/P)^{u/k-1}.$$

On substituting this relation into (5.6) and then recalling (5.4) and (5.5), therefore, we obtain

$$\begin{aligned}\mathfrak{R}_u(m) - \mathfrak{S}_u(m;P)I(m) &\leqslant \sum_{1\leqslant q\leqslant P} |I_u(m;q)-I(m)| \sum_{\substack{a=1\\(a,q)=1}}^{q} |q^{-1}S(q,a)|^u \\ &\ll (n/P)^{u/k-1} V_1^{P+1}(u;u/k-1).\end{aligned}$$

Our hypothesis concerning u ensures that $u/k - 1 \geqslant J + (2+\delta_k)/k$, and thus we discern from Lemma 5.1 that

$$\mathfrak{R}_u(m) - \mathfrak{S}_u(m;P)I(m) \ll P^{u-k-J-1/(2k)}. \tag{5.8}$$

The integral (5.7) is the familiar singular integral in Waring's problem. In the integral form (3.1) in which we have defined the generating function $v(\beta)$, a classical treatment of the type described on [2, pp. 21–23] yields the formula

$$I(m) = \Delta_m \frac{\Gamma(1+1/k)^u}{\Gamma(u/k)} m^{u/k-1}.$$

Also, our hypothesis on u leads from (1.2) and (5.5) via (5.4) and Lemma 5.1 to the bound

$$\mathfrak{S}_u(m) - \mathfrak{S}_u(m;P) \leqslant V_P^{\infty}(u;0) \ll P^{1-(u-1-\delta_k)/k+\varepsilon} \ll P^{-J-1/(2k)}.$$

The proof of the lemma follows by substituting these estimates into (5.8). □

This lemma may be combined with Lemma 4.4 in order to obtain an asymptotic formula for $\mathfrak{K}_{u,l}(n)$.

Lemma 5.3. *Suppose that u is an integer with $u \geqslant (J+1)k + 2 + \delta_k$. Then there is a positive number η for which*

$$\mathfrak{K}_{u,l}(n) = 2^{u+l} \frac{\Gamma(1+1/k)^{u+l}}{\Gamma((u+l)/k)} \mathfrak{S}_{u+l}(n) n^{(u+l)/k-1} + O\big(P^{u+l-k-J-\eta}\big).$$

Proof. On recalling formula (5.2) for $\mathfrak{K}_{u,l}(n)$, we find from Lemma 5.2 that there is a positive number η such that

$$\mathfrak{K}_{u,l}(n) = 2^u \frac{\Gamma(1+1/k)^u}{\Gamma(u/k)} T_1 + O(P^{u+l-k-J-\eta}), \tag{5.9}$$

where

$$T_1 = \sum_{\substack{|m_1|\leqslant P \\ m_1^k+\cdots+m_l^k\leqslant n}} \cdots \sum_{|m_l|\leqslant P} \mathfrak{S}_u(n-m_1^k-\cdots-m_l^k)(n-m_1^k-\cdots-m_l^k)^{u/k-1}.$$

Applying the definition (1.2) of the singular series, we find that

$$T_1 = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} (q^{-1}S(q,a))^u \Omega(n;q,a), \tag{5.10}$$

where $\Omega(n;q,a)$ is equal to

$$\sum_{\substack{|m_1|\leqslant P \\ m_1^k+\cdots+m_l^k\leqslant n}} \cdots \sum_{|m_l|\leqslant P} (n-m_1^k-\cdots-m_l^k)^{u/k-1} e(-(n-m_1^k-\cdots-m_l^k)a/q).$$

We sort the summands into arithmetic progressions modulo q and recall (4.8). Thus we see that

$$\Omega(n;q,a) = \sum_{r_1=1}^{q} \cdots \sum_{r_l=1}^{q} \Xi_{q,\mathbf{r}}^{(l)}(P;u/k-1)e(-(n-r_1^k-\cdots-r_l^k)a/q).$$

When $1\leqslant q\leqslant P$, we apply Lemma 4.4 with $N=J+1$ to obtain

$$\Omega(n;q,a) = 2^l \frac{\Gamma(u/k)\Gamma(1+1/k)^l}{\Gamma((u+l)/k)} n^{(u+l)/k-1} T_2 + O(T_3), \tag{5.11}$$

where

$$T_2 = q^{-l} \sum_{r_1=1}^{q} \cdots \sum_{r_l=1}^{q} e(-(n-r_1^k-\cdots-r_l^k)a/q)$$

and

$$T_3 = q^l P^{u-k}(q/P)^{J+1-l} \ll q^{J+1/(2k)} P^{u+l-k-J-1/(2k)}.$$

When $q>P$, meanwhile, one has the trivial estimate $\Omega(n;q,a) \ll P^{u+l-k}$. On recalling (1.7), we find that $T_2 = (q^{-1}S(q,a))^l e(-na/q)$. Thus, on substituting (5.11) into (5.10) and recalling (5.4) and (5.5), we discern that

$$T_1 = 2^l \frac{\Gamma(u/k)\Gamma(1+1/k)^l}{\Gamma((u+l)/k)} \mathfrak{S}_{u+l}(n;P) n^{(u+l)/k-1} + O(T_4), \tag{5.12}$$

where

$$T_4 = P^{u+l-k-J-1/(2k)} V_1^P(u;J+1/(2k)) + P^{u+l-k} V_P^{\infty}(u;0).$$

In view of our hypothesis on u, an application of Lemma 5.1 yields the bound

$$T_4 \ll P^{u+l-k-J-1/(2k)}.$$

Then on recalling the first conclusion of Lemma 5.2, we deduce from (5.12) that

$$T_1 = 2^l \frac{\Gamma(u/k)\Gamma(1+1/k)^l}{\Gamma((u+l)/k)} \mathfrak{S}_{u+l}(n) n^{(u+l)/k-1} + O(P^{u+l-k-J-1/(2k)}).$$

Making use of this estimate within (5.9), therefore, we obtain the asymptotic formula claimed in the statement of the lemma, and thus the proof of the lemma is complete. □

6. Combining the major arc contributions, for even k

Having evaluated asymptotically the expression $\mathfrak{K}_{u,l}(n)$, under appropriate conditions on u, we next seek to assemble the contributions comprising $\mathfrak{I}_{t,l}(n)$, and thereby evaluate the number $R_s(n)$.

Lemma 6.1. *When l and t are natural numbers with $t - l \geqslant (J+1)k + 2 + \delta_k$, one has*

$$\mathfrak{I}_{t,l}(n) = o\big(P^{t-k-J}\big).$$

Meanwhile, one has

$$\mathfrak{I}_{t,0}(n) = 2^t \frac{\Gamma(1+1/k)^t}{\Gamma(t/k)} n^{t/k-1} \mathfrak{S}_t(n) + o\big(P^{t-k-J}\big).$$

Proof. It follows from (3.6), (5.1) and the binomial theorem that

$$\begin{aligned}\mathfrak{I}_{t,l}(n) &= \sum_{v=0}^{l} (-1)^v \binom{l}{v} \int_{\mathfrak{M}} (2f^*(\alpha))^{t-l+v} h(\alpha)^{l-v} e(-n\alpha)\,\mathrm{d}\alpha \\ &= \sum_{v=0}^{l} (-1)^v \binom{l}{v} \mathfrak{K}_{t-l+v,l-v}(n).\end{aligned}$$

Suppose temporarily that $l \geqslant 1$. Then we find from Lemma 5.3 that

$$\mathfrak{I}_{t,l}(n) = 2^t \frac{\Gamma(1+1/k)^t}{\Gamma(t/k)} \mathfrak{S}_t(n) n^{t/k-1} \sum_{v=0}^{l} (-1)^v \binom{l}{v} + o\big(P^{t-k-J}\big).$$

The first conclusion of the lemma consequently follows by noting that

$$\sum_{v=0}^{l} (-1)^v \binom{l}{v} = (1-1)^l = 0 \quad (l \geqslant 1).$$

When $l = 0$, meanwhile, the desired conclusion follows directly from Lemma 5.2. This completes the proof of the lemma. □

We are now equipped to prove Theorem 1.1. Suppose that s is J-admissible for k. Observe first that, as a consequence of Lemma 6.1, one finds that whenever $0 \leqslant r \leqslant J$ and $s - 2J \geqslant (J+1)k + 2 + \delta_k$, then

$$\sum_{l=0}^{2J-2r} \binom{s-r}{l} \mathfrak{I}_{s-r,l}(n) = 2^{s-r} \frac{\Gamma(1+1/k)^{s-r}}{\Gamma((s-r)/k)} \mathfrak{S}_{s-r}(n) n^{(s-r)/k-1} + o\big(P^{s-k-J}\big).$$

We therefore deduce from (3.9) that whenever $s \geqslant (J+1)(k+2) + \delta_k$, then

$$R_s(n) = \sum_{r=0}^{J} \binom{s}{r} \left(-\tfrac{1}{2}\right)^r \frac{\Gamma(1+1/k)^{s-r}}{\Gamma((s-r)/k)} \mathfrak{S}_{s-r}(n) n^{(s-r)/k-1} + o\big(n^{(s-J)/k-1}\big).$$

On recalling (1.5) and (1.6), we find that the proof of Theorem 1.1 is complete.

7. Preliminary manœuvres, for odd k

Our approach to proving Theorem 1.2 is broadly similar to that employed in the proof of Theorem 1.1. Although we are consequently able to economise in our exposition, numerous technical complications force us to discuss this odd situation separately. We now suppose that s and k are natural numbers with $s > k \geqslant 3$ and k odd. On this occasion we consider directly the number $R_s(n)$ of integral representations of n in the shape (2.1) with $1 \leqslant x_i \leqslant P$ $(1 \leqslant i \leqslant s)$. When $\mathfrak{B}$ is measurable, we put

$$R_s(n;\mathfrak{B}) = \int_{\mathfrak{B}} f(\alpha)^s e(-n\alpha)\,\mathrm{d}\alpha. \tag{7.1}$$

Thus, by orthogonality, one has

$$R_s(n) = R_s(n;\mathfrak{M}) + R_s(n;\mathfrak{m}). \tag{7.2}$$

Lemma 7.1. *When s is J-admissible for k, one has*

$$R_s(n;\mathfrak{m}) = o(P^{s-k-J}).$$

Proof. On applying the triangle inequality, the desired conclusion is immediate from the definition of a J-admissible exponent. □

8. The major arc contribution truncated, for odd k

Recalling the definition (3.2) of $f^*(\alpha)$, an application of the binomial theorem within (7.1) reveals that

$$R_s(n;\mathfrak{M}) = \sum_{l=0}^{s} \binom{s}{l} \mathfrak{I}^{\dagger}_{s,l}(n), \tag{8.1}$$

where

$$\mathfrak{I}^{\dagger}_{s,l}(n) = \int_{\mathfrak{M}} f^*(\alpha)^{s-l}(f(\alpha)-f^*(\alpha))^l e(-n\alpha)\,\mathrm{d}\alpha. \tag{8.2}$$

Lemma 8.1. *Suppose that J is a non-negative integer. Then whenever $l > 2J$ and $s \geqslant \max\{l, k+2J+4\}$, one has*

$$\mathfrak{I}^{\dagger}_{s,l}(n) = o(P^{s-k-J}).$$

Proof. The argument of the proof of Lemma 3.1 applies, mutatis mutandis, to confirm the conclusion of the lemma by noting (3.3). □

When $s \geqslant k+2J+4$, the conclusion of Lemma 8.1 combines with (7.2), Lemma 7.1 and (8.1) to deliver the formula

$$R_s(n) = \sum_{l=0}^{2J} \binom{s}{l} \mathfrak{I}^{\dagger}_{s,l}(n) + o\big(P^{s-k-J}\big). \tag{8.3}$$

9. An auxiliary lemma, for odd k

We now apply Lemmata 4.1 and 4.2 to obtain an asymptotic formula for an auxiliary sum of use for odd k. Let X be a large positive real number, and let θ be a non-negative real number. When $q \in \mathbb{N}$ and $r \in \mathbb{Z}$, we define

$$\Upsilon^{\dagger}_{q,r}(X;\theta) = \sum_{-r/q<h\leqslant(X-r)/q} \big(X^k-(qh+r)^k\big)^{\theta}. \tag{9.1}$$

Lemma 9.1. *When $1 \leqslant N \leqslant \lceil\theta\rceil$, one has*

$$\Upsilon^{\dagger}_{q,r}(X;\theta) = q^{-1}X^{k\theta+1}\frac{\Gamma(1+\theta)\Gamma(1+1/k)}{\Gamma(1+\theta+1/k)} + \Psi + O\big(X^{k\theta}(q/X)^{N-1}\big),$$

where

$$\Psi = X^{k\theta}\sum_{0\leqslant\nu\leqslant(N-1)/k}\frac{\Gamma(1+\theta)}{\nu!(\nu k+1)\Gamma(1+\theta-\nu)}\beta_{\nu k+1}(-r/q)(q/X)^{k\nu}.$$

Proof. We apply Lemma 4.2 with F and G given by formula (4.2). Write $a=-r/q$ and $b=(X-r)/q$. Then one finds that $G(a)=X^k$ and $G(b)=0$. Moreover, since formula (4.3) remains valid, one has $G^{(j)}(a)=0$ for $1\leqslant j<k$, and also for $j>k$, and $G^{(k)}(a)=-k!q^k$. Formula (4.4) also remains valid. From Lemma 4.2, we thus deduce that $F(G(x))$ has continuous derivatives through the N-th order on (a,b), continuous derivatives through the $(N-1)$-st order on $[a,b]$, and further $|\mathrm{d}^N F(G(x))/\mathrm{d}x^N|$ is integrable on $[a,b]$. Note also that when $0\leqslant m<\theta$, one has $F^{(m)}(G(b))=0$, and hence

$$\frac{\mathrm{d}^\kappa}{\mathrm{d}x^\kappa}F(G(x))\Big|_{x=b}=0 \quad (0\leqslant\kappa<\theta).$$

In addition, it follows from Lemma 4.2 that

$$\frac{\mathrm{d}^\kappa}{\mathrm{d}x^\kappa}F(G(x))\Big|_{x=a}=0 \quad (0\leqslant\kappa<\theta),$$

except possibly when κ is divisible by k, say $\kappa=\nu k$, in which case

$$\begin{aligned}\frac{\mathrm{d}^{\nu k}}{\mathrm{d}x^{\nu k}}F(G(x))\Big|_{x=a} &= \frac{(\nu k)!}{\nu!}F^{(\nu)}(G(a))\bigg(\frac{G^{(k)}(a)}{k!}\bigg)^{\nu}\\ &= \frac{(\nu k)!}{\nu!}\theta(\theta-1)\dots(\theta-\nu+1)X^{k\theta-k\nu}(-1)^{\nu}q^{k\nu}.\end{aligned}$$

On substituting these values into Lemma 4.1, we see that

$$\begin{aligned}\sum_{a<h\leqslant b}F(G(h)) = \int_a^b F(G(x))\,\mathrm{d}x - \sum_{0\leqslant\nu\leqslant(N-1)/k}T_\nu\\ - \frac{(-1)^N}{N!}\int_a^b \beta_N(x)\frac{\mathrm{d}^N}{\mathrm{d}x^N}F(G(x))\,\mathrm{d}x,\end{aligned} \tag{9.2}$$

where

$$T_\nu = \frac{(-1)^{\nu k+1+\nu}}{(\nu k+1)!}\frac{(\nu k)!}{\nu!}\frac{\Gamma(1+\theta)}{\Gamma(1+\theta-\nu)}X^{k(\theta-\nu)}q^{k\nu}\beta_{\nu k+1}(a).$$

By making the change of variable $y=(qx+r)/X$, we find that

$$\int_a^b F(G(x))\,\mathrm{d}x = q^{-1}X^{k\theta+1}\int_0^1 (1-y^k)^\theta\,\mathrm{d}y = q^{-1}X^{k\theta+1}\frac{\Gamma(1+\theta)\Gamma(1+1/k)}{\Gamma(1+\theta+1/k)}. \tag{9.3}$$

Also, just as in the corresponding treatment described in the argument of the proof of Lemma 4.3, one finds that

$$\int_a^b \beta_N(x)\frac{\mathrm{d}^N}{\mathrm{d}x^N}F(G(x))\,\mathrm{d}x \ll X^{k\theta}(q/X)^{N-1}.$$

On recalling that k is odd and $a=-r/q$, the conclusion of the lemma follows on substituting this estimate together with (9.3) into formula (9.2), and then recalling the definition (9.1) of $\Upsilon_{q,r}^{\dagger}(X;\theta)$. □

We extend the previous conclusion so as to handle a multidimensional generalisation. When $q\in\mathbb{N}$ and $r_1,\dots,r_l\in\mathbb{Z}$, we define

$$\Xi_{q,\mathbf{r}}^{\dagger(l)}(X;\theta)=\sum_{\substack{0<x_1\leqslant X\\ x_1\equiv r_1 \pmod{q}}}\cdots\sum_{\substack{0<x_l\leqslant X\\ x_l\equiv r_l \pmod{q}}}(X^k-x_1^k-\dots-x_l^k)^\theta, \tag{9.4}$$

where the summands are constrained by the inequality $x_1^k+\dots+x_l^k\leqslant X^k$. It is convenient also to introduce a multidimensional analogue of the Bernoulli polynomials specific to the purpose at hand. Let $\sigma_m(y_1,\dots,y_l)$ denote the m-th elementary symmetric polynomial in $y_1,\dots,y_l$, and define

$$B_m^{(l)}(q;\mathbf{r})=\sigma_m(\beta_1(-r_1/q),\dots,\beta_1(-r_l/q)).$$

Note that $\sigma_0(y_1,\dots,y_l)=1$, and by convention $\sigma_{-1}(y_1,\dots,y_l)=0$.

Lemma 9.2. *When $1\leqslant N\leqslant\min\{\lceil\theta\rceil,k+1\}$, one has*

$$\Xi_{q,\mathbf{r}}^{\dagger(l)}(X;\theta)=X^{k\theta}\sum_{m=0}^{l}\frac{\Gamma(1+\theta)\Gamma(1+1/k)^{l-m}}{\Gamma(1+\theta+(l-m)/k)}B_m^{(l)}(q;\mathbf{r})(X/q)^{l-m} + O\big(X^{k\theta}(q/X)^{N-1}(1+X/q)^{l-1}\big).$$

Proof. We proceed by induction on l, noting that the case $l=1$ is already a consequence of Lemma 9.1. Suppose then that $L>1$, and that the desired conclusion has been established for $1\leqslant l<L$. From (9.4), we obtain

$$\Xi_{q,\mathbf{r}}^{\dagger(l)}(X;\theta)=\sum_{-r_L/q<h_L\leqslant(X-r_L)/q}\Xi_{q,\mathbf{r}'}^{\dagger(L-1)}(Y;\theta), \tag{9.5}$$

where

$$\mathbf{r}'=(r_1,\dots,r_{L-1})\quad\text{and}\quad Y=\big(X^k-(qh_L+r_L)^k\big)^{1/k}.$$

Our inductive hypothesis delivers the asymptotic formula

$$\Xi_{q,\mathbf{r}'}^{\dagger(L-1)}(Y;\theta)=Y^{k\theta}\sum_{m=0}^{L-1}\frac{\Gamma(1+\theta)\Gamma(1+1/k)^{L-m-1}}{\Gamma(1+\theta+(L-m-1)/k)}B_m^{(L-1)}(q;\mathbf{r}')(Y/q)^{L-m-1} + O\big(Y^{k\theta}(q/Y)^{N-1}(1+Y/q)^{L-2}\big).$$

By substituting this expression into (9.5), we deduce that

$$\Xi^{\dagger(L)}_{q,\mathbf{r}}(X;\theta) = \sum_{m=0}^{L-1} \frac{\Gamma(1+\theta)\Gamma(1+1/k)^{L-m-1}}{\Gamma(1+\theta+(L-m-1)/k)} B^{(L-1)}_m(q;\mathbf{r}')q^{m+1-L}T_m + O\big(X^{k\theta}(q/X)^{N-1}(1+X/q)^{L-1}\big), \tag{9.6}$$

where

$$T_m = \sum_{-r_L/q<h_L\leqslant (X-r_L)/q} \big(X^k-(qh_L+r_L)^k\big)^{\theta+(L-m-1)/k}. \tag{9.7}$$

An application of Lemma 9.1 leads from (9.7) to the asymptotic formula

$$T_m = q^{-1}X^{k\theta+L-m}\frac{\Gamma(1+\theta+(L-m-1)/k)\Gamma(1+1/k)}{\Gamma(1+\theta+(L-m)/k)} + X^{k\theta+L-m-1}\beta_1(-r_L/q) + O\big(X^{k\theta+L-m-1}(q/X)^{N-1}\big),$$

whence from (9.6) one obtains the relation

$$\Xi^{\dagger(L)}_{q,\mathbf{r}}(X;\theta) = X^{k\theta}\sum_{m=0}^{L} \frac{\Gamma(1+\theta)\Gamma(1+1/k)^{L-m}}{\Gamma(1+\theta+(L-m)/k)} C_m(q,\mathbf{r})(X/q)^{L-m} + O\big(X^{k\theta}(q/X)^{N-1}(1+X/q)^{L-1}\big),$$

where

$$C_m(q,\mathbf{r}) = B^{(L-1)}_m(q;\mathbf{r}') + \beta_1(-r_L/q)B^{(L-1)}_{m-1}(q;\mathbf{r}').$$

By considering the relevant symmetric polynomials, one sees that $C_m(q,\mathbf{r}) = B^{(L)}_m(q;\mathbf{r})$. Thus we conclude that the inductive hypothesis holds for $l=L$, confirming the inductive step and completing the proof of the lemma. □

In Lemma 9.2 we have limited the parameter N to be at most $k+1$ in order that terms involving $\beta_{\nu k+1}(-r_i/q)$ with $\nu \geqslant 1$ be absent. A more detailed investigation reveals that such additional terms can be accommodated at the expense of substantial complications.

10. The major arc contribution evaluated, for odd *k*

We turn next to the evaluation of the integral $\mathfrak{I}^\dagger_{s,l}(n)$ defined in (8.2). With this objective in mind, we consider the auxiliary integral

$$\mathfrak{K}^\dagger_{u,l}(n) = \int_{\mathfrak{M}} f^*(\alpha)^u f(\alpha)^l e(-n\alpha)\,\mathrm{d}\alpha. \tag{10.1}$$

Making use of the definition (1.3) of $f(\alpha)$, and recalling (5.3), one finds that

$$\mathfrak{K}^\dagger_{u,l}(n) = \sum_{1\leqslant m_1\leqslant P}\cdots\sum_{1\leqslant m_l\leqslant P} \mathfrak{R}_u(n-m_1^k-\cdots-m_l^k). \tag{10.2}$$

Recall the exponential sum $T(q,a)$ defined in (1.7), and the modified singular series $\mathfrak{S}_{s,j}(n)$ defined in (1.8). It is useful also to define the truncation

$$\mathfrak{S}_{s,j}(n;Q) = \sum_{1\leqslant q\leqslant Q}\sum_{\substack{a=1\\(a,q)=1}}^{q} (q^{-1}S(q,a))^{s-j}T(q,a)^j e(-na/q). \tag{10.3}$$

These modified singular series have good convergence properties, as a consequence of the following simple estimate for $T(q,a)$.

Lemma 10.1. *Suppose that $a \in \mathbb{Z}$ and $q \in \mathbb{Z}$ satisfy $(q,a) = 1$. Then for each $\varepsilon > 0$, one has $T(q,a) \ll q^{1/2+\varepsilon}$.*

Proof. We find from (1.7) that

$$T(q,a) = -\tfrac{1}{2}S(q,a) + q^{-1}\sum_{r=1}^{q}\int_r^q e(ar^k/q)\,\mathrm{d}x. \tag{10.4}$$

By interchanging the order of summation and integration here, we deduce from [13, equation (4.14)] that

$$\begin{aligned} T(q,a) + \tfrac{1}{2}S(q,a) &= q^{-1}\int_0^q \sum_{1\leqslant r\leqslant x} e(ar^k/q)\,\mathrm{d}x \\ &= q^{-1}\int_0^q \big(q^{-1}S(q,a)x + O\big(q^{1/2+\varepsilon}\big)\big)\,\mathrm{d}x \\ &= \tfrac{1}{2}S(q,a) + O\big(q^{1/2+\varepsilon}\big), \end{aligned}$$

and the lemma follows at once. □

Lemma 10.2. *When $t > k + r + 1$, one has*

$$\mathfrak{S}_{t,r}(n;Q) \ll 1 + \big(Q^{1/k}\big)^{r(1+k/2)+k-(t-3/2)}.$$

Moreover, whenever $t \geqslant \frac{1}{2}(r+2)(k+2)$, the modified singular series $\mathfrak{S}_{t,r}(n)$ is absolutely convergent.

Proof. On recalling (5.4) and (10.3), one finds from Lemmata 5.1 and 10.1 that

$$\mathfrak{S}_{t,r}(n;Q) \ll V_1^{Q+1}(t-r; r/2+\varepsilon) \ll 1 + \big(Q^{1/k}\big)^{(r+2)(k+2)/2-t-1/2}.$$

This confirms the first assertion of the lemma. The second follows on observing that the hypothesis $t \geqslant \frac{1}{2}(r+2)(k+2)$ ensures in like manner that

$$\sum_{1\leqslant q\leqslant Q}\ \sum_{\substack{a=1\\(a,q)=1}}^{q} |q^{-1}S(q,a)|^{t-r}|T(q,a)|^r \ll V_1^{Q+1}(t-r; r/2+\varepsilon) \ll 1. \qquad \square$$

We are now equipped to evaluate $\mathfrak{K}^{\dagger}_{u,l}(n)$.

Lemma 10.3. *Suppose that J and u are non-negative integers satisfying $J \leqslant k$ and $u \geqslant (J+1)k + 2$. Then there is a positive number η for which*

$$\mathfrak{K}^{\dagger}_{u,l}(n) = n^{(u+l)/k-1}\sum_{m=0}^{\min\{l,J\}} \mathfrak{D}_m n^{-m/k} + O\big(P^{u+l-k-J-\eta}\big),$$

where

$$\mathfrak{D}_m = \binom{l}{m}\frac{\Gamma(1+1/k)^{u+l-m}}{\Gamma((u+l-m)/k)}\mathfrak{S}_{u+l,m}(n).$$

Proof. On recalling formula (10.2) for $\mathfrak{R}^{\dagger}_{u,l}(n)$, we find from Lemma 5.2 that there is a positive number η such that

$$\mathfrak{R}^{\dagger}_{u,l}(n) = \frac{\Gamma(1+1/k)^u}{\Gamma(u/k)} T_1 + O\big(P^{u+l-k-J-\eta}\big), \tag{10.5}$$

where

$$T_1 = \sum_{\substack{1 \leqslant m_1 \leqslant P \\ m_1^k + \cdots + m_l^k \leqslant n}} \cdots \sum_{1 \leqslant m_l \leqslant P} \mathfrak{S}_u(n - m_1^k - \cdots - m_l^k)(n - m_1^k - \cdots - m_l^k)^{u/k-1}.$$

Applying the definition (1.2) of the singular series, we find that

$$T_1 = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} (q^{-1}S(q,a))^u \Omega^{\dagger}(n;q,a), \tag{10.6}$$

where $\Omega^{\dagger}(n;q,a)$ is equal to

$$\sum_{\substack{1 \leqslant m_1 \leqslant P \\ m_1^k + \cdots + m_l^k \leqslant n}} \cdots \sum_{1 \leqslant m_l \leqslant P} (n - m_1^k - \cdots - m_l^k)^{u/k-1} e(-(n - m_1^k - \cdots - m_l^k)/q).$$

By sorting summands into arithmetic progressions modulo q and recalling (9.4), we see that

$$\Omega^{\dagger}(n;q,a) = \sum_{r_1=1}^{q} \cdots \sum_{r_l=1}^{q} \Xi^{\dagger(l)}_{q,\mathbf{r}}(P; u/k-1) e(-(n - r_1^k - \cdots - r_l^k)a/q).$$

When $1 \leqslant q \leqslant P$, we apply Lemma 9.2 with $N = J+1$, obtaining

$$\Omega^{\dagger}(n;q,a) = n^{u/k-1} \sum_{m=0}^{l} \frac{\Gamma(u/k)\Gamma(1+1/k)^{l-m}}{\Gamma((u+l-m)/k)} n^{(l-m)/k} U_m + O(U^*), \tag{10.7}$$

where

$$U_m = q^{m-l} \sum_{r_1=1}^{q} \cdots \sum_{r_l=1}^{q} B_m^{(l)}(q;\mathbf{r}) e(-(n - r_1^k - \cdots - r_l^k)a/q)$$

and

$$U^* = q^l P^{u-k}(q/P)^{J+1-l} \ll q^{J+1/(2k)} P^{u+l-k-J-1/(2k)}.$$

When $q > P$, meanwhile, one has the trivial estimate $\Omega^{\dagger}(n;q,a) \ll P^{u+l-k}$. On recalling formula (1.7), we find that

$$U_m = \binom{l}{m} (q^{-1}S(q,a))^{l-m} T(q,a)^m e(-na/q).$$

Thus, on substituting (10.7) into (10.6) and recalling (5.4) and (10.3), we discern that

$$T_1 = n^{u/k-1} \sum_{m=0}^{l} \binom{l}{m} \frac{\Gamma(u/k)\Gamma(1+1/k)^{l-m}}{\Gamma((u+l-m)/k)} \mathfrak{S}_{u+l,m}(n;P) n^{(l-m)/k} + O(T_2), \tag{10.8}$$

where

$$T_2 = P^{u+l-k-J-1/(2k)} V_1^P(u; J+1/(2k)) + P^{u+l-k} V_P^{\infty}(u;0).$$

In view of our hypothesis on u, an application of Lemma 5.1 delivers the bound

$$T_2 \ll P^{u+l-k-J-1/(2k)}.$$

In addition, by applying Lemma 5.1 via (5.4) to (1.8) and (10.3), one discerns that our hypothesis on u ensures that when $0 \leqslant m \leqslant J$, one has

$$\mathfrak{S}_{u+l,m}(n) - \mathfrak{S}_{u+l,m}(n;P) \ll V_P^{\infty}(u+l-m; m/2+\varepsilon) \ll P^{m-J-1/(2k)}.$$

Meanwhile, when $m > J$, it follows from Lemma 10.2 that

$$\mathfrak{S}_{u+l,m}(n;P) \ll 1 + P^{m+1-(u+l-3/2)/k} \ll P^{m-J-1/(2k)}.$$

On substituting these estimates into (10.8), we conclude that

$$\begin{aligned} T_1 - n^{(u+l)/k-1} \sum_{m=0}^{\min\{l,J\}} \binom{l}{m} \frac{\Gamma(u/k)\Gamma(1+1/k)^{l-m}}{\Gamma((u+l-m)/k)} n^{-m/k} \mathfrak{S}_{u+l,m}(n) \\ \ll P^{u-k} \sum_{m=0}^{l} P^{l-m}(P^{m-J-1/(2k)}) \ll P^{u+l-k-J-1/(2k)}. \end{aligned}$$

The conclusion of the lemma now follows from (10.5). □

11. Combining the major arc contributions, for odd *k*

We now reassemble the integrals $\mathfrak{I}^{\dagger}_{s,l}(n)$ so as to evaluate $R_s(n)$.

Lemma 11.1. *Suppose that $0 \leqslant J \leqslant k$, and that l and s are natural numbers with $s - l \geqslant (J+1)k + 2$. Then whenever $l > J$, one has*

$$\mathfrak{I}^{\dagger}_{s,l}(n) = o\big(P^{s-k-J}\big).$$

Meanwhile, when instead $l \leqslant J$, one has

$$\mathfrak{I}^{\dagger}_{s,l}(n) = \frac{\Gamma(1+1/k)^{s-l}}{\Gamma((s-l)/k)} \mathfrak{S}_{s,l}(n) n^{(s-l)/k-1} + o\big(P^{s-k-J}\big).$$

Proof. It follows from (8.2), (10.1) and the binomial theorem that

$$\begin{aligned} \mathfrak{I}^{\dagger}_{s,l}(n) &= \sum_{v=0}^{l} (-1)^v \binom{l}{v} \int_{\mathfrak{M}} f^*(\alpha)^{s-l+v} f(\alpha)^{l-v} e(-n\alpha)\, d\alpha \\ &= \sum_{v=0}^{l} (-1)^v \binom{l}{v} \mathfrak{K}^{\dagger}_{s-l+v,l-v}(n). \end{aligned}$$

Then we find from Lemma 10.3 that

$$\mathfrak{I}^{\dagger}_{s,l}(n) = \sum_{v=0}^{l} (-1)^v \binom{l}{v} \sum_{m=0}^{\min\{l-v,J\}} \binom{l-v}{m} \mathfrak{B}_m + o\big(P^{s-k-J}\big),$$

where

$$\mathfrak{B}_m = \frac{\Gamma(1+1/k)^{s-m}}{\Gamma((s-m)/k)} \mathfrak{S}_{s,m}(n) n^{(s-m)/k-1}.$$

On making use of the identity

$$\binom{l}{v}\binom{l-v}{m}=\binom{l}{m}\binom{l-m}{v},$$

therefore, we deduce that

$$\mathfrak{I}^{\dagger}_{s,l}(n)=\sum_{m=0}^{\min\{l,J\}}\binom{l}{m}\sum_{v=0}^{l-m}(-1)^v\binom{l-m}{v}\mathfrak{B}_m+o\big(P^{s-k-J}\big). \tag{11.1}$$

When $m<l$, one has

$$\sum_{v=0}^{l-m}(-1)^v\binom{l-m}{v}=(1-1)^{l-m}=0,$$

so that only the terms with $m=l$ contribute in (11.1). Thus we see that when $l>J$, the outermost sum on the right-hand side of (11.1) contributes nothing, and this confirms the first conclusion of the lemma. When $l\leqslant J$, meanwhile, the only contribution comes from those terms with $m=l$ and $v=0$, and in this way one obtains the second conclusion of the lemma. □

We are now equipped to prove Theorem 1.2. Suppose that $0\leqslant J\leqslant k$, and that s is J-admissible for k. Observe first that, as a consequence of Lemma 11.1, one finds that whenever $s-2J\geqslant (J+1)k+2$, then

$$\sum_{l=0}^{2J}\binom{s}{l}\mathfrak{I}^{\dagger}_{s,l}(n)=\sum_{l=0}^{J}\binom{s}{l}\frac{\Gamma(1+1/k)^{s-l}}{\Gamma((s-l)/k)}\mathfrak{S}_{s,l}(n)n^{(s-l)/k-1}+o\big(P^{s-k-J}\big).$$

We therefore deduce from (8.3) that whenever $s\geqslant (J+1)(k+2)$, then

$$R_s(n)=\sum_{l=0}^{J}\binom{s}{l}\frac{\Gamma(1+1/k)^{s-l}}{\Gamma((s-l)/k)}\mathfrak{S}_{s,l}(n)n^{(s-l)/k-1}+o\big(n^{(s-J)/k-1}\big). \tag{11.2}$$

On recalling (1.5) and (1.9), we find that the proof of Theorem 1.2 is complete.

We have yet to discuss the modified singular series $\mathfrak{S}_{s,l}(n)$. It is clear, however, that the limitation $J\leqslant k$ can be removed if one is prepared to endure further analysis in which exponential sums of the shape

$$\sum_{r=1}^{q}\beta_{k+1}(-r/q)e(ar^k/q),$$

and yet more exotic creatures, appear. The complexity rises rapidly, and we avoid discussion of such matters in the absence of deserving applications.

Notice also that when k is even, one has

$$T(q,a)=\sum_{r=1}^{q}\Big(\frac12-\frac{r}{q}\Big)e(ar^k/q)=\sum_{r=0}^{q-1}\Big(\frac12-\frac{q-r}{q}\Big)e(ar^k/q),$$

so that $T(q,a)=-1-T(q,a)$. We therefore see that when k is even, one has $T(q,a)=-\frac12$, and hence it follows from (1.8) that $\mathfrak{S}_{s,j}(n)=(-\frac12)^j\mathfrak{S}_{s-j}(n)$. The asymptotic formula (11.2) is therefore consistent with that delivered by Theorem 1.1, at least in those restricted circumstances where $J\leqslant k$.

12. Exceptional sets

Our goal in this section is to establish Theorem 1.3. We take an abbreviated approach, concentrating on the contribution of the minor arcs. Let N be a large positive number, and put $P = N^{1/k}$. We assume that the exponent $2s$ is $2J$-admissible for k. Thus, for some positive monotonic function $L(t)$ growing sufficiently slowly in terms of t, and with $L(t) \to +\infty$ as $t \to \infty$, one has

$$\int_{\mathfrak{m}} |f(\alpha)|^{2s}\, d\alpha \ll P^{2s-k-2J} L(P)^{-3}.$$

Define the function $F(\alpha)$ by taking $F(\alpha) = f(\alpha)^s$ when $\alpha \in \mathfrak{m}$, and otherwise by taking $F(\alpha) = 0$. Also, let $\hat{F}(n)$ be the Fourier coefficient of F, so that

$$\hat{F}(n) = \int_0^1 F(\alpha) e(-\alpha n)\, d\alpha = \int_{\mathfrak{m}} f(\alpha)^s e(-n\alpha)\, d\alpha.$$

Then by Bessel's inequality, one has

$$\sum_{n\in\mathbb{Z}} |\hat{F}(n)|^2 \leqslant \int_{\mathfrak{m}} |f(\alpha)|^{2s}\, d\alpha \ll P^{2s-k-2J} L(P)^{-3}. \tag{12.1}$$

Let $\mathcal{Z}_s(N)$ denote the set of integers n with $N/2 < n \leqslant N$ for which one has

$$|\hat{F}(n)| > P^{s-k-J} L(P)^{-1}.$$

We write Z for $\mathrm{card}(\mathcal{Z}_s(N))$, and note that there is no loss of generality in supposing that $L(t)$ grows no more rapidly than $\log t$. Then it is immediate from the relation (12.1) that $Z \ll P^k L(P)^{-1}$, and thus we see that for all but at most $O(NL(N^{1/k})^{-1})$ integers n with $N/2 < n \leqslant N$, one has

$$\int_{\mathfrak{m}} f(\alpha)^s e(-n\alpha)\, d\alpha = o\big(P^{s-k-J}\big).$$

Of course, with only modest adjustments in this argument, one may show that for all but at most $O(NL(N^{1/k})^{-1})$ integers n with $N/2 < n \leqslant N$, one has likewise

$$\int_{\mathfrak{m}} h(\alpha)^s e(-n\alpha)\, d\alpha = o\big(P^{s-k-J}\big).$$

The corresponding major arc contributions

$$\int_{\mathfrak{M}} f(\alpha)^s e(-n\alpha)\, d\alpha \quad \text{and} \quad \int_{\mathfrak{M}} h(\alpha)^s e(-n\alpha)\, d\alpha$$

are obtained by means of the work of Sections 2–11, with inconsequential modification. Thus the conclusions of Theorem 1.3 follow from the work that we have already completed, after summing over dyadic intervals. One has merely to note that the integers n with $1 \leqslant n \leqslant \sqrt{N}$ make a negligible contribution to the exceptional set for which the desired asymptotic formula fails to hold, and the exceptional set arising from those integers n with $\sqrt{N} < n \leqslant N$ is no larger than

$$\sum_{\substack{m=0\\ 2^m \leqslant \sqrt{N}}}^{\infty} \mathrm{card}(\mathcal{Z}_s(2^{-m}N)) \ll L\big(N^{1/(2k)}\big)^{-1} \sum_{m=0}^{\infty} 2^{-m} N \ll NL\big(N^{1/(2k)}\big)^{-1}.$$

13. The modified singular series $\mathfrak{S}_{s,r}(n)$

It remains to discuss the modified singular series $\mathfrak{S}_{s,r}(n)$. These series are presumably non-zero in general. Such is the case when k is even, for then, as we have noted at the end of Section 11, one has

$$\mathfrak{S}_{s,r}(n) = \left(-\tfrac{1}{2}\right)^r \mathfrak{S}_{s-r}(n),$$

and under modest local conditions, the conventional singular series $\mathfrak{S}_{s-r}(n)$ is indeed non-zero. When k is odd, however, the situation is less clear.

We spend some time now examining the situation in which k is odd, beginning with the proof of Theorem 1.4. Write

$$T^\dagger(q,a) = \sum_{r=0}^{q} \left(\frac{1}{2} - \frac{r}{q}\right) e(ar^k/q).$$

We begin by noting that

$$T(q,a) = -\tfrac{1}{2} + T^\dagger(q,a),$$

whence

$$\overline{T^\dagger(q,a)} = T^\dagger(q,-a) = \sum_{r=0}^{q} \left(\frac{1}{2} - \frac{r}{q}\right) e(a(q-r)^k/q) = -T^\dagger(q,a).$$

It follows that $T^\dagger(q,a)$ is purely imaginary. Observe that when $s \geqslant \frac{3}{2}(k+2)$, so that both $\mathfrak{S}_{s,1}(n)$ and $\mathfrak{S}_{s-1}(n)$ are absolutely convergent, one has

$$\begin{aligned}
\mathfrak{S}_{s,1}(n) + \tfrac{1}{2}\mathfrak{S}_{s-1}(n) &= \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} (q^{-1}S(q,a))^{s-1}\left(\tfrac{1}{2} + T(q,a)\right)e(-na/q) \\
&= \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} (q^{-1}S(q,a))^{s-1} T^\dagger(q,a) e(-na/q).
\end{aligned}$$

In present circumstances, where k is odd, one has $S(q,a) = S(q,-a)$, and thus we are led from our earlier discussion to the interim conclusion

$$\begin{aligned}
\mathfrak{S}_{s,1}(n) + \tfrac{1}{2}\mathfrak{S}_{s-1}(n) &= \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} (q^{-1}S(q,-a))^{s-1} T^\dagger(q,-a) e(na/q) \\
&= -\sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} (q^{-1}S(q,a))^{s-1} T^\dagger(q,a) e(na/q) \\
&= -\mathfrak{S}_{s,1}(-n) - \tfrac{1}{2}\mathfrak{S}_{s-1}(-n).
\end{aligned}$$

The relation $S(q,a) = S(q,-a)$ similarly ensures that $\mathfrak{S}_{s-1}(n) = \mathfrak{S}_{s-1}(-n)$, and hence we conclude that

$$\mathfrak{S}_{s,1}(n) + \mathfrak{S}_{s,1}(-n) = -\mathfrak{S}_{s-1}(n). \tag{13.1}$$

Observe next that when $s \geqslant \frac{3}{2}k + 3$, then it follows from Lemma 10.1 via (5.4) and Lemma 5.1 that

$$\mathfrak{S}_{s,1}(n) - \mathfrak{S}_{s,1}(n; Q) \leqslant V_Q^\infty\big(s-1; \tfrac{1}{2}+\varepsilon\big) \ll Q^{-1/(2k)}.$$

Note also that when Q is a natural number and n is a multiple of $Q!$, then for $1 \leqslant q \leqslant Q$ one has $e(-na/q) = 1 = e(na/q)$. Thus, with the same assumptions on n, one has

$$\begin{aligned}
\mathfrak{S}_{s,1}(n) &= \sum_{1\leqslant q\leqslant Q} \sum_{\substack{a=1\\(a,q)=1}}^{q} (q^{-1}S(q,a))^{s-1}T(q,a)e(-na/q) + O\big(Q^{-1/(2k)}\big)\\
&= \sum_{1\leqslant q\leqslant Q} \sum_{\substack{a=1\\(a,q)=1}}^{q} (q^{-1}S(q,a))^{s-1}T(q,a)e(na/q) + O\big(Q^{-1/(2k)}\big)\\
&= \mathfrak{S}_{s,1}(-n) + O\big(Q^{-1/(2k)}\big).
\end{aligned}$$

Thus we deduce from (13.1) that

$$\mathfrak{S}_{s,1}(n) = -\tfrac{1}{2}\mathfrak{S}_{s-1}(n) + O\big(Q^{-1/(2k)}\big).$$

This confirms the conclusion of Theorem 1.4.

When $r \geqslant 2$, the behaviour of $\mathfrak{S}_{s,r}(n)$ is less clear, since $T^\dagger(q,a)^r$ is real whenever r is even, and the above device fails. However, some information concerning non-vanishing of linear combinations of the series $\mathfrak{S}_{s,r}(n)$ would be available with additional work.

We finish by proving Theorem 1.5. Suppose that $s \geqslant \frac{1}{2}(r+2)(k+2)$. Then the conclusion of Lemma 10.2 shows that the modified singular series $\mathfrak{S}_s(n;r)$ is absolutely convergent. Thus there is a constant c_r for which $|\mathfrak{S}_{s,r}(n)| \leqslant c_r$. For the sake of concision, write

$$U(q,a) = S(q,a)^{s-r}T(q,a)^r.$$

Let η be a positive number with $\eta < c_r$, and choose Q in such a way that

$$\sum_{q\geqslant Q} \sum_{\substack{a=1\\(a,q)=1}}^{q} q^{r-s}|U(q,a)| < \eta.$$

Then from (1.8) one discerns that

$$\sum_{1\leqslant n\leqslant x} |\mathfrak{S}_{s,r}(n)|^2 \geqslant \sum_{1\leqslant n\leqslant x} \Bigg| \sum_{1\leqslant q<Q} \sum_{\substack{a=1\\(a,q)=1}}^{q} q^{r-s}U(q,a)e(-na/q)\Bigg|^2 - 3c_r\eta x.$$

On squaring out the sums over q and a, the diagonal contribution is

$$T_1 = \lfloor x\rfloor \sum_{1\leqslant q<Q} \sum_{\substack{a=1\\(a,q)=1}}^{q} q^{2r-2s}|U(q,a)|^2,$$

whilst the off-diagonal terms make a contribution

$$T_2 \ll \sum_{1\leqslant q<Q} \sum_{\substack{a=1\\(a,q)=1}}^{q} q^{r+1-s}|U(q,a)| \sum_{1\leqslant w<Q} \sum_{\substack{b=1\\(b,w)=1\\b/w\neq a/q}}^{w} w^{r+1-s}|U(w,b)|.$$

On recalling (5.4), we find from Lemma 10.1 that

$$T_2 \ll \Bigg(\sum_{1 \leqslant q < Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} q^{r+1-s} |U(q,a)| \Bigg)^2 \ll V_1^Q\big(s-r; \tfrac{1}{2}r + 1 + \varepsilon\big)^2.$$

Hence, provided that $s \geqslant \frac{1}{2}(r+4)(k+2)$, we may apply Lemma 5.1 to deduce that $T_2 \ll 1$. Meanwhile, in a similar fashion, one sees that under the same conditions on s, one has

$$\sum_{1 \leqslant q < Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} q^{2r-2s} |U(q,a)|^2 \ll V_1^Q(2s-2r; r+\varepsilon) \ll 1.$$

Hence the sum on the left-hand side here converges as $Q \to \infty$. Since this series is clearly positive, it follows that for some $\delta > 0$ one has $T_1 \geqslant 4\delta^2 x$. We now fix η with $0 < \eta < \delta^2/(3c_r)$, and conclude that

$$\sum_{1 \leqslant n \leqslant x} |\mathfrak{S}_{s,r}(n)|^2 \geqslant 4\delta^2 x - 3c_r \eta x + O(1) > 2\delta^2 x.$$

The last sum is the key to our proof of Theorem 1.5. The number of natural numbers n with $1 \leqslant n \leqslant x$ for which $|\mathfrak{S}_{s,r}(n)| \geqslant \delta$ is at least

$$\begin{aligned} c_r^{-2} \sum_{\substack{1 \leqslant n \leqslant x \\ |\mathfrak{S}_{s,r}(n)| \geqslant \delta}} |\mathfrak{S}_{s,r}(n)|^2 &\geqslant c_r^{-2} \Bigg(\sum_{1 \leqslant n \leqslant x} |\mathfrak{S}_{s,r}(n)|^2 - \delta^2 x \Bigg) \\ &\geqslant c_r^{-2}(2\delta^2 x - \delta^2 x) > (\delta/c_r)^2 x. \end{aligned}$$

Thus we conclude that $|\mathfrak{S}_s(n;r)| \geqslant \delta$ for a positive proportion of the integers n with $1 \leqslant n \leqslant x$, and this completes the proof of Theorem 1.5.

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